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THE CROSSING NUMBER OF THE CONE OF A GRAPH*

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Abstract. Motivated by a problem asked by Richter and by the long standing Harary–Hill conjecture, we study the relation between the crossing number of a graph G and the crossing number of its cone CG, the graph obtained from G by adding a new vertex adjacent to all the vertices in G. Simple examples show that the difference $cr(CG) - cr(G)$ can be arbitrarily large for any fixed $k = cr(G)$. In this work, we are interested in finding the smallest possible difference; that is, for each nonnegative integer k, find the smallest $f(k)$ for which there exists a graph with crossing number at least k and cone with crossing number $f(k)$. For small values of k, we give exact values of $f(k)$ when the problem is restricted to simple graphs and show that $f(k) = k + \Theta(\sqrt{k})$ when multiple edges are allowed.

Key words. math, crossing number, cone, graph, combinatorics

AMS subject classifications. 68R10, 05C62

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1. Introduction.

Little is known about the relation between the crossing number and the chromatic number. Albertson's conjecture [2], whose motivation is to understand more about this relation, states that if $\chi(G) \geq r$, then $cr(G) \geq cr(K_r)$. This conjecture has been proved [2, 5, 18] for $r \leq 16$. It is related to Hajós' conjecture stating that every r-chromatic graph contains a subdivision of K_r. If G contains a subdivision of K_r, then $cr(G) \geq cr(K_r)$. Thus Albertson's conjecture is weaker than Hajós' conjecture; however Hajós' conjecture is false for each $r \geq 7$ [9].

The *cone* of a graph G is the graph CG obtained from G by adding an *apex*, a new vertex that is adjacent to each vertex in G. Many properties of a graph automatically transfer to its cone. For example, if G is r-coloring-critical, then CG is $(r + 1)$-coloring-critical. During the Crossing Numbers Workshop in 2013, in an attempt to understand Alberston's conjecture, Richter proposed the following problem: given an integer $n \geq 5$ and a graph G with crossing number at least $cr(K_n)$, does it follow that the crossing number of its cone CG is at least $cr(K_{n+1})$?

The answer to Richter's question is positive for the first interesting case when $n = 5$: Kuratowski's theorem implies that the cone of every graph with crossing number at least $cr(K_5) = 1$ contains a subdivision of CK_5 or $CK_{3,3}$. It is not hard to see that each of these two graphs, $CK_5 = K_6$ and $CK_{3,3}$, has crossing number equal to $3 = cr(K_6)$. Unfortunately, the answer is negative for the next case, as the graph in Figure 1 shows. This graph has crossing number 3 and a cone with crossing

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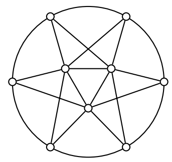

FIG. 1. *A counterexample to Richter's question when* $n = 6$.

number at most 6, and this is less than $cr(K_7) = 9$. This motivated us to investigate the following question.

PROBLEM 1. *For each* $k \geq 0$*, find the smallest integer* $f(k)$ *for which there is a graph* G *with crossing number at least* k *and whose cone has* $cr(CG) = f(k)$.

Note that $f(k)$ can also be defined as the largest integer such that every graph with $cr(G) \geq k$ has $cr(CG) \geq f(k)$. There are examples in which the values of $cr(G)$ and $cr(CG)$ can differ arbitrarily (for instance, if G is the disjoint union of K_4's and K_5's). What is less clear is how close these values can be.

An upper bound to the function $f(k)$ is obtained from the graph in Figure 1 by changing the multiplicity of each edge to r. Each drawing of the new graph has at least $3r^2$ crossings, and its cone has crossing number at most $3r^2 + 3r$. This shows that $f(k) \leq k + \sqrt{3k}$. Our main result shows that this is close to the best possible.

THEOREM 2. *Let* G *be a graph with* $cr(G) \geq k$*. Then* $cr(CG) \geq k + \sqrt{k/2}$.

Thus we have the following.

COROLLARY 3. *For multigraphs we have* $f(k) = k + \Theta(\sqrt{k}\,)$.

The paper is organized as follows. Page drawings, a concept intimately related to drawings of the cone of a graph, are defined in section 2 and used throughout the subsequent sections. Although there seems to be a connection between 1-page drawings and drawings of the cone, their exact relationship is subtle. Our proofs are instructive in this manner and provide further understanding of these concepts.

The proof of our main result, Theorem 2, is provided in section 3. In section 4, we restrict Problem 1 to the case of simple graphs. To distinguish between these two problems, we use $f_s(k)$ instead of $f(k)$. In this paper, a graph is allowed to have multiple edges but no loops; when our graphs have no multiple edges, then we refer to them as simple graphs. We find the smallest values of f_s by showing that $f_s(1) = 3$, $f_s(2) = 5$, $f_s(3) = 6$, $f_s(4) = 8$, and $f_s(5) = 10$. These initial values may suggest that $f_s(k) \geq 2k$. However, in section 5 we show that

$$f_s(k) = k + o(k)$$

and provide additional justification for a more specific conjecture that

$$f_s(k) = k + \sqrt{2}\,k^{3/4}(1 + o(1)).$$

As we are interested in drawings minimizing the number of crossings, we may assume henceforth that in all our drawings (a) any two edges have at most one crossing in common, (b) no two incident edges cross each other, and (c) no three edges share a common crossing point.

2. Book drawings. In this section we describe a perspective provided by considering *book drawings* of graphs, a concept that has been studied for its own sake and has interesting applications. Our goal is to prove Corollary 6, a key tool used in the proofs of Theorems 2 and 7. For a more detailed discussion of book drawings see [4, 8, 17].

For an integer $k \geq 1$, a *k-page book* consists of k half planes sharing their boundary line ℓ (spine). A *k-page drawing* is a drawing of a graph in which vertices are placed on the spine of a k-page book, and each edge arc is contained in one page. A convenient way to visualize a k-page drawing is by means of the *circular model*. In this model each page is represented by a 2-dimensional unit disk, so that the vertices are arranged identically on each disk boundary and each edge is drawn entirely in exactly one disk. In this work we are only interested in 1- and 2-page drawings, and, to be more precise, in the following problem.

PROBLEM 4. *Given a* 1*-page drawing of a graph G with k crossings, find an upper bound on the number of crossings of an optimal* 2*-page drawing of G while keeping the order of vertices of G on the spine.*

In other words, if the drawing of G in the plane is such that all the vertices are incident to the outer-face (which is equivalent to having a 1-page drawing), what is the most efficient way to redraw some edges in the outer-face to reduce the number of crossings? For this purpose, we define the *circle graph* C_D of a 1-page drawing D of G as the graph whose vertices are the edges of G and in which two elements are adjacent if and only if they cross in D. Note that C_D depends only on the cyclic order of the vertices of G on the spine.

A related problem was previously formulated by Kainen in [15], where he studied the *outer-planar crossing number* of a graph as the minimum number of crossings in any drawing of G so that all its vertices are incident to the same face. Clearly, the crossing number of CG is at most the outer-planar crossing number of G. Although Kainen was interested in finding an n-vertex graph that has the largest difference between its crossing number and its outer-planar crossing number, for us it will be useful to consider drawings in which most of the vertices are incident to the same face.

Turning a 1-page drawing into a 2-page drawing is equivalent to finding a bipartition $(X, V(C_D)\setminus X)$ of the vertices of C_D, each part representing the set of edges of G drawn in one of the pages. Minimizing the number of crossings in the obtained 2-page drawing of G is equivalent to maximizing the number of edges in C_D between X and $V(C_D)\setminus X$. The latter problem is known as the *max-cut problem*. If the graph C_D has m edges, then a well-known result of Erdős [10] states that its maximum edge cut has size more than $m/2$. Improvements to this general bound are known (see [11, 12] and a more recent survey [6]). For our purpose the following bound of Edwards will be useful.

LEMMA 5 (Edwards [11, 12]). *Suppose that G is a graph of order n with $m \geq 1$ edges. Then G contains a bipartite subgraph with at least $\frac{1}{2}m + \sqrt{\frac{1}{8}m + \frac{1}{64}} - \frac{1}{8} > \frac{1}{2}m$ edges.*

In our context, this result translates to the following observation that we will use.

COROLLARY 6. *Let D be a* 1*-page drawing of a graph G with $k \geq 1$ crossings. Then some edges of G can be redrawn in a new page, obtaining a* 2*-page drawing with at most $\frac{1}{2}k - \sqrt{\frac{1}{8}k + \frac{1}{64}} + \frac{1}{8}$ crossings. Such a drawing can be found in time $O(|E(G)| + k)$.*

Proof. We can turn a 1-page drawing D with k crossings into a 2-page drawing as follows. We define the circle graph C_D whose vertices are edges of G and whose edges correspond to crossings in D. By taking a bipartition of $V(C_D)$ containing at least

$\frac{1}{2}k + \sqrt{\frac{1}{8}k + \frac{1}{64}} - \frac{1}{8}$ edges and drawing the edges of G corresponding to one part of the bipartition in the second page, we obtain a 2-page drawing as claimed. Such a partition of $V(C_D)$ exists by Lemma 5 and can be found in time $O(|E(G)| + k)$ by the results of Bollobás and Scott [6, Theorem 22]. ∎

3. Lower bound on the crossing number of the cone. This section contains the proof of our main result.

Proof of Theorem 2. Let $\widehat{D}$ be an optimal drawing of the cone CG of G with apex a, and suppose $\widehat{D}$ has less than $k + \sqrt{k/2}$ crossings. We consider $D = \widehat{D}|_G$, the drawing of G induced by $\widehat{D}$. If we let t be the number of crossings in D, then we have

$$k \le t < k + \sqrt{k/2}. \tag{3.1}$$

For each vertex $v \in V(G) \cup \{a\}$, let s_v be the number of crossings in $\widehat{D}$ involving edges incident with v. We may assume that, for each $v \in V(G)$, $s_v < \sqrt{k/2}$. Otherwise, by removing v and all the edges incident to v, we obtain a drawing of $CG - v$ containing a subdrawing of G, in which v is represented by the apex, and this drawing has less than k crossings, a contradiction.

Consider $x_1,\dots,x_{s_a}$, the crossings involving edges incident with a. Since $\widehat{D}$ is optimal, each of these crossings is between an edge incident to a and an edge in G. Let $e_1,\dots,e_{s_a}$ be the list of edges in G (we allow repetitions) so that x_i is the crossing between e_i and an edge incident with a. We subdivide each edge e_i in D using two points close to the crossing x_i, and we remove the edge segment σ_i joining these new two vertices, in order to obtain a drawing D_0 of a graph G_0 with t crossings (see Figure 2).

In the drawing D_0 all vertices, including the subdivision vertices, are incident to the face of D_0 containing the point corresponding to the apex vertex a of CG in $\widehat{D}$. For simplicity, we may assume that this is the unbounded face of D_0. It follows that there exists a simple closed curve ℓ in the closure of this face, containing all the vertices of G_0. Thus, D_0 gives rise to a 1-page drawing of G_0 with spine ℓ.

Now construct a new drawing of G as follows:

Step 1. Start with the 1-page drawing D_0. Partition the edges of G_0 according to Corollary 6, and draw the edges of one part in page 2 outside ℓ.

Step 2. Reinsert edge segments $\sigma_1, \dots, \sigma_{s_a}$ as they were drawn in D to obtain a drawing D_1 (of a subdivision) of G. These segments do not cross each other, but they may cross some edges of G_0 that we placed in page 2 in Step 1.

Now we estimate the number of crossings in D_1. According to Corollary 6, after Step 1 we obtain a 2-page drawing D_0 with less than $t/2 - \sqrt{t/8} + 1/8$ crossings. After

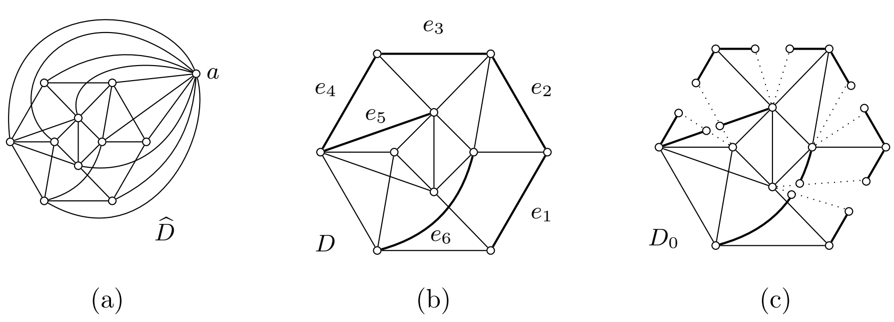

FIG. 2. *A drawing where the crossed edges are cut.*

Step 2 we gain some new crossings between the added segments $\sigma_1, \dots, \sigma_{s_a}$ and the edges of G_0 drawn on page 2 in Step 1.

Claim. We can choose the routings of the edges of G_0 drawn in page 2 such that the number of new crossings between $\sigma_1, \dots, \sigma_{s_a}$ and the edges drawn on page 2 in Step 1 is at most $(k-1)/2$.

Proof. Let $e \in E(G)$ be an edge having ends u, $v \in V(G)$. Suppose that $ay_1, \dots, ay_{r_e}$ are the edges incident to a that cross e in $\widehat{D}$. We may assume that, for every i, j with $1 \le i < j \le r_e$, when we traverse e from u to v, the crossing $x_i = e \cap ay_i$ precedes the crossing $x_j = e \cap ay_j$. It is convenient to let $x_0 = u$ and $x_{r_e+1} = v$.

The edges of G_0 included in $D[e]$ are the segments of $D[e] - \{\sigma_1, \dots, \sigma_{s_a}\}$. We enumerate these edges as $\tau_0^e, \dots, \tau_{r_e}^e$ so that τ_i^e is included in the $x_i x_{i+1}$-arc of $D[e]$. Note that τ_0^e is incident to u, while $\tau_{r_e}^e$ is incident to v.

Let $T = \{\tau_i^e : e \in E(G) \text{ and } 0 \le i \le r_e\}$ be the set of edges of G_0. In Step 1, when we apply Corollary 6 to the edges in D_0, we obtain a partition $T_1 \cup T_2$ of T. Instead of counting how many crossings are between the segments in $\sigma_1, \dots, \sigma_{s_a}$ and the edges in one of the T_i's when we redraw T_i in page 2, we estimate the number m of crossings between $\sigma_1, \dots, \sigma_{s_a}$ and the edges in T when we draw all the crossing edges in T in page 2. This will show that one of the two parts, either T_1 or T_2, can be drawn in page 2 creating at most $m/2$ crossings with the segments $\sigma_1, \dots, \sigma_{s_a}$. To show our claim, it suffices to prove that $m \le k-1$, and this is what we do next.

For every point p distinct from a and contained in an edge f incident to a, the *depth* $h(p)$ of p is the number of crossings in $\widehat{D}$ contained in the open subarc of f connecting a to p. When we redraw an edge τ_i^e in page 2, we can draw it so that it crosses at most $h(x_i) + h(x_{i+1})$ segments in $\sigma_1, \dots, \sigma_{s_a}$. Such a new drawing of τ_i^e is obtained from letting the segment of τ_i^e near to x_i follow the same dual path in D that x_i follows to reach a via ay_i. Likewise the new end of τ_i^e near x_{i+1} is defined. The new τ_i^e is obtained from connecting the two *end segments* of τ_i^e inside the face of D containing a.

Let $X(a)$ be the set of crossings involving edges incident to a in D. For every $x \in X(a)$, there are precisely two elements in T, so that when they are redrawn in page 2, one of its end segments mimics the arc between x and a inside the edge including x and a. Each $v \in V(G)$ is incident to at most s_v edges crossing in D_0. Then, for every $v \in V(G)$, there are are most s_v edges in T, so that when we redraw them in page 2, one of their ends mimics the dual path followed by the edge $\widehat{D}[va]$. These two observations together imply that

$$\begin{aligned} m &\le \sum_{x \in X(a)} 2h(x) + \sum_{v \in V} h(v) s_v \\ &< 2 \sum_{v \in V} (1 + 2 + \ldots + (h(v) - 1)) + \sqrt{k/2} \sum_{v \in V} h(v) \\ &\le \sum_{v \in V} h(v)^2 + (\sqrt{k/2}) s_a \le \left(\sum_{v \in V} h(v) \right)^2 + k/2 \\ &= s_a^2 + k/2 < k. \end{aligned}$$

Because m is an integer less than k, $m \le k-1$ as desired. ∎

At the end, we obtained a drawing D_1 of (a subdivision of) G with less than $t/2 - \sqrt{t/8} + 1/8 + (k-1)/2$ crossings. Using (3.1) it follows that

$$cr(D_1) < \frac{1}{2}(k + \sqrt{k/2}) - \sqrt{t/8} + 1/8 + k/2 - 1/2 = k + \sqrt{k/8} - \sqrt{t/8} - 3/8 < k,$$

contradicting the fact that $cr(D_1) \geq cr(G) \geq k$. ◻

4. Exact values of the crossing number of the cone for simple graphs. In this section, we investigate the minimum crossing number of a cone, with the restriction of only considering simple graphs. We are interested in finding the smallest integer $f_s(k)$ for which there is a simple graph with crossing number at least k, whose cone has crossing number $f_s(k)$. On one hand, we describe below a family of simple graphs that shows that $f_s(k) \leq 2k$. Our best general lower bound is obtained from Theorem 2. The main result in this section, Theorem 7, helps us to obtain exact values of $f_s(k)$ for cases when k is small.

THEOREM 7. *Let G be a simple graph with crossing number k. Then*
(1) *if $k \geq 2$, then $cr(CG) \geq k + 3$;*
(2) *if $k \geq 4$, then $cr(CG) \geq k + 4$; and*
(3) *if $k \geq 5$, then $cr(CG) \geq k + 5$.*

Before proving Theorem 7, we describe a family of examples that is used to find an upper bound for $f_s(k)$, which is exact for the values $k = 3, 4, 5$. Given an integer $k \geq 3$, the graph F_k (Figure 3) is obtained from two disjoint cycles $C_1 = x_0 \dots x_{k-1}x_0$ and $C_2 = y_0 \dots y_{2k-1}y_0$ by adding, for each $i = 0, \dots, k-1$, the edges x_iy_{2i-2}, x_iy_{2i-1}, x_iy_{2i}, x_iy_{2i+1} (where the indices of the vertices y_j are taken modulo $2k$). It is not hard to see that F_k has crossing number k: a drawing with k crossings is shown in Figure 3. To show that $cr(F_k) \geq k$, for $i \in \{0, \dots, k-1\}$, consider L^i, the K_4 induced by the vertices in $\{x_i, x_{i+1}, y_{2i}, y_{2i+1}\}$. Every L^i is a subgraph of a K_5 subdivision of F_k; thus, in an optimal drawing of F_k, at least one of the edges in L^i is crossed. This only guarantees that $cr(F_k) \geq k/2$, as two edges from distinct L^i's might be crossed. However, if an edge from L^i crosses an edge e_j from some other L^j, then $F_k - e_j$ has a K_5 subdivision including L^i, exhibiting a new crossing in some edge in L^i. Therefore, every L^i either has a crossing not involving an edge in another L^j, or there are at least two crossings involving edges in L^i. This shows that $cr(F_k) \geq k$.

The graph shown in Figure 4 has crossing number 2, and its cone has crossing number at most 5. This shows that $f_s(2) \leq 5$. On the other hand, F_3, F_4, and F_5 serve as examples to show that $f_s(k) \leq 2k$ for $k = 3, 4, 5$. These bounds are tight for $2 \leq k \leq 5$ by Theorem 7.

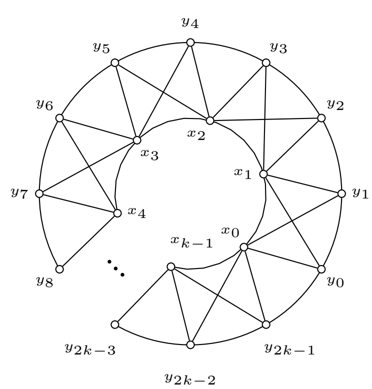

FIG. 3. *The graph F_k.*

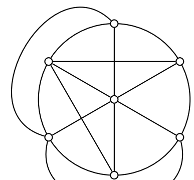

FIG. 4. *A graph with crossing number* 2 *whose cone has crossing number* 5.

Proof of Theorem 7. Suppose G is a graph with $cr(G) = k$. Let $\widehat{D}$ be an optimal drawing of the cone CG, D its restriction to G, and F_a be the face of D containing the apex a. The vertices of G incident to F_a are the *planar neighbors* of a.

Assume that $k \geq 2$, and suppose $\widehat{D}$ has exactly $k + t$ crossings. Theorem 2 guarantees that $t \geq 1$. Since each edge from a to a nonplanar neighbor introduces at least one crossing, the apex a has either 0, 1, 2, 3, or 4 nonplanar neighbors (if a has more than 4 nonplanar neighbors, then all items in Theorem 7 are satisfied).

We start by assuming that a has no nonplanar neighbors. In this case, D is a 1-page drawing of G. Corollary 6 implies that we can obtain a new drawing of G with less than $(k + t)/2$ crossings. Thus $(k + t)/2 > cr(G) = k$, which implies that $t \geq k + 1$. In each case of the theorem, this implies the conclusion; thus we may now assume that a has at least one and at most t nonplanar neighbors.

(1) Let us now assume that $k \geq 2$ and $t \leq 2$.

Suppose a has exactly one nonplanar neighbor u. Then D has at most $k + 1$ crossings. At least one edge incident to u is crossed in D, otherwise, all the crossed edges have ends in F_a, and using Corollary 6, we obtain a drawing of G with less than $(k + 1)/2$ crossings, contradicting that $cr(G) = k$. If at least two crossings in D involve edges incident to u, or if D has k crossings, then we obtain a drawing of G with less than k crossings by redrawing u as the point representing a in $\hat{D}$ and adding all the edges from u to each neighbor v of u, by following the corresponding edge arc connecting a to v in $\hat{D}$. Therefore D has $k + 1$ crossings, and exactly k of them involve edges not incident to u. Again, we apply Corollary 6, but this time we are more careful by setting our two pages in such a way that the edge not incident to u that crosses an edge incident to u is redrawn in the page contained in F_a. In this case we obtain a drawing of G with less than $\frac{k}{2}$ crossings (note that if we apply Corollary 6 without the additional assumption, then we only guarantee a drawing of G with less than $k/2 + 1$ crossings).

Finally, suppose a has exactly two distinct nonplanar neighbors u and v. Then, $\widehat{D}$ has $k + 2$ crossings, D has k crossings, and the edges au, av are crossed exactly once. Notice that each crossed edge in D is incident to either u or v; otherwise, we can redraw such an edge inside F_a, obtaining a drawing of G with less than k crossings. Redraw v as the point representing a in $\widehat{D}$, draw the edge uv (if it exists in G) as the edge au in $\widehat{D}$, and draw the edges from v to its neighbors distinct from u, inside F_a without creating new crossings. Since every crossing in D involves an edge incident with v, we obtain a drawing of G with at most one crossing, a contradiction.

On a side note, our last argument requires G to be simple: it fails when there are k parallel edges between u and v. The assumption of G being simple is also used in similar arguments of (2) and (3), when a is replaced by a nonplanar neighbor of a.

(2) Now, suppose that $k \geq 4$ and that $t = 3$.

The case when the apex a has only one nonplanar neighbor u is similar to the argument in (1). If at least three crossings in D involve edges incident with u, then by redrawing u and the edges incident to u in F_a, we obtain a drawing with less than k crossings, a contradiction. Thus, at most two crossings involve edges incident to u. We redraw the remaining crossed edges according to Corollary 6 (if there is an edge e that crosses an edge incident to u, in order to remove an extra crossing, we may choose this new drawing so that e is redrawn in the page contained in F_a). If exactly two crossings involve edges incident to u, then the resulting drawing has at most $\frac{k}{2}+1$ crossings, where the $+1$ is due to the fact that e was redrawn in the page contained in F_a. If at most one of the edges at u is crossed, then the new drawing has at most $(k+1)/2$ crossings. In either case, since $k \geq 4$, the new drawing has less than k crossings, a contradiction.

Let us now consider the case when the apex has two nonplanar neighbors u and v. In this case, the drawing D has either k or $k+1$ crossings, and one of $\{au, av\}$, say au, is crossed only once. Let L be the set of crossed edges in D that are not incident to u or v. Suppose there are at least two crossings involving only edges in L. Then, either there are two edges in L that do not cross, or L has an edge e that crosses two other edges in L. In the former case, we redraw such a pair of edges in F_a; in the latter case, we redraw e in F_a. Both of these modifications yield a drawing with less than k crossings. Thus, we may assume that at most one crossing in D involves two edges not incident to u or v. Redraw v as the point representing a in $\widehat{D}$, draw the edge vu (if this edge exists in G) as au, and redraw the remaining edges from v to its neighbors distinct from u without creating new crossings. The new drawing of G has at most two crossings: one, possibly, along av and another between two edges in L, a contradiction.

Finally suppose that the apex a has three nonplanar neighbors u, v, w. In this case D has precisely k crossings, and the edges au, av, aw are crossed exactly once. Observe that any crossed edge in D is incident to one of $\{u, v, w\}$; otherwise we can redraw such an edge in F_a, obtaining a drawing of G with less than k crossings.

Let H be the graph induced by $\{u, v, w\}$. For $x \in \{u, v, w\}$, let $d_H(x)$ denote the degree of x in H. Then at most $d_H(x)$ crossings in $\hat{D}$ involve edges at x. Otherwise, we can redraw x as a and connect x to its neighbors by following the corresponding edges incident to a.

This gives us a drawing of G with less than k crossings. So for each vertex $x \in \{u, v, w\}$, there are at most two crossings involving edges at x. Hence D has at most three crossings, a contradiction.

(3) Now, suppose that $k \geq 5$ and that $t = 4$.

Let N denote the set of nonplanar neighbors of a. For $u \in N$, let s_{au} denote the number crossings involving the edge au in $\widehat{D}$, and let $s = \sum_{u \in N} s_{au}$. As we did before, we distinguish cases depending on the size of N. We showed, before proving item (1), that $|N| \geq 1$. We need the following observation.

Claim. If $u \in N$, then the following holds:

(i) At most $4 - s_{au}$ crossings in D involve edges incident to u.

(ii) The number of crossings in which both edges involved are incident to some vertex in N is at most $2|N| - \lceil s/2 \rceil$.

Proof. (i) If there are more than $4 - s_{au}$ crossings involving edges at u, then we redraw u as a and join u to its neighbors using the corresponding edges from a to $V(G)$. This is a drawing with less than

$$k + 4 - s - (4 - s_{au}) + (s - s_{au}) = k$$

crossings, a contradiction.

(ii) From (i), we know that for each $u \in N$, there are at most $4 - s_{au}$ crossings in D involving edges at u. Let us count the number of pairs (u, x) where $u \in N$ and x is a crossing involving an edge incident with u. By (i), the number of such pairs is at most

$$\sum_{u \in N} (4 - s_{au}) = 4|N| - s. \tag{4.1}$$

This implies (ii).

Case 1. The apex a has exactly one planar neighbor.

From item (i) in the claim, we know that there are at most $4 - s_{au}$ crossings involving edges incident with $u \in N$. So at least $k + 4 - (4 - s_{au}) - s_{au} = k$ crossings involve crossing pairs that are not incident to u. We apply Corollary 6 to redraw some crossing edges not incident with u in F_a, and we are careful by choosing our two pages so that we draw one edge crossing an edge at u (if such an edge exists) in F_a to remove an extra crossing. We obtain a drawing with at most $k + 4 - s_{au} - (\frac{k}{2} + \frac{1}{2}) - 1 < k$ crossings.

Let us define $r = cr(D) - k$ for brevity. Note that $r = 4 - s$.

Case 2. The apex a has two nonplanar neighbors.

In this case $0 \leq r \leq 2$. Let X be the set of crossings involving an edge none of whose ends is in N, and let E_X be the set of crossing edges having both ends not in N. We claim that $|X| \leq r$.

This is easy to see when $r = 0$; if $|X| \geq 1$, then there is an edge in E_X that we can redraw in F_a to obtain a drawing of G with less than k crossings. Suppose that $r = 1$. If $|X| \geq 2$, then either there is an edge $e \in E_X$ including two crossings, or there is a pair of edges in E_X that do not cross each other in D. In the former case we redraw e in F_a; in the latter we redraw the pair in F_a. In each case, the edges that are drawn in F_a have no crossings in the new drawing because the order of the endpoints along the boundary of F_a determines whether two edges cross. In both cases we obtain a drawing of G with less than k crossings.

Finally, suppose that $r = 2$. Any edge in E_X is involved in at most two crossings; otherwise we could redraw it in F_a to obtain a drawing of G with less than k crossings. If $|X| \geq 3$, then either there is an edge $e \in E_X$ crossed twice and an edge $f \in E_X$ not crossing e, or every edge in E_X is crossed at most once. In the former case, we redraw e and f in F_a; in the latter, for each crossing in X we pick an edge E_X involved in the crossing and redraw it in F_a. In either case we obtain a drawing of G with less than k crossings. Therefore $|X| \leq 2$.

We conclude that $|X| \leq r \leq 2$. By item (ii), the number of crossings in D is at most

$$4 - \left\lceil \frac{4 - r}{2} \right\rceil + |X| \leq 2 + r/2 + |X| \leq 2 + 3r/2 \leq 5.$$

Since $cr(G) \geq 5$, we have that $cr(D) = 5$, $|X| = r = 2$, and $E_X \neq \emptyset$. However, if we redraw any edge from E_X in F_a, we obtain a drawing of G with less than five crossings.

Case 3. The apex a has three nonplanar neighbors.

In this case $cr(D)$ is either k or $k + 1$, so $r = 0$ or $r = 1$. The argument given in the previous case shows that there are at most r crossings involving an edge with both ends not in N. Using item (ii), we obtain that D has at most $4 + r/2 + |X| \leq 4 + r/2 + r \leq 5 + 1/2$ crossings, where X is defined as in the previous case. Since $cr(D) \geq 5$, this shows that D has exactly five crossings and that $|X| = 1$. In particular $cr(D) = k$ and thus $r = 0$, which contradicts that $|X| \leq r$.

Case 4. The apex has four nonplanar neighbors.

In this case $s = 4$, $cr(D) = k$, and $s_{ay} = 1$ for every $y \in N$. Let $N = \{u, v, w, x\}$. Note that each crossing edge is incident to a vertex in N, because otherwise we could reroute one of the edges through F_a and reduce the number of crossings. By item (i), there are at most three crossings involving edges incident to a fixed vertex in N, and by (ii), $cr(D) \leq 6$. Moreover, the count (4.1) in the proof of (ii) shows that $cr(D) = 5$ if there is an edge with both ends in N that is involved in a crossing.

Let H be the graph induced by N. We distinguish two cases depending on whether $D[H]$ is a crossing K_4 or not.

Subcase 1. $D[H]$ is not a crossing K_4.

If $H = K_4$, then $D[H]$ is a planar K_4. This implies that there is a 3-cycle composed of vertices in N, separating a fourth vertex in N from a, and this contradicts that $s_{ay} = 1$ for every $y \in N$. Therefore, there is a pair of vertices in H; say u and v, with $uv \notin E(G)$.

Recall that, for $y \in N$, $d_H(y)$ denotes the degree of y in H, then at most $d_H(y)$ crossings involve edges at y. Otherwise, by redrawing y in $\widehat{D}[a]$, drawing the edges from y to its neighbors in H by using the respective edges from a; and drawing the remaining edges at y in F_a without creating new crossings, we obtain a drawing of G with less than k crossings. Since $d_H(u), d_H(v) \leq 2$ and $d_H(w)$, $d_H(x) \leq 3$, $cr(D)$ is at most $(\sum_{y \in N} d_H(y))/2 = 5$. Because $cr(D) \geq 5$, this implies that $cr(D) = 5$, $d_H(u) = d_H(v) = 2$, and $d_H(w) = d_H(x) = 3$. This also shows that for each $y \in N$, the number of crossings in D involving edges at y is exactly $d_H(y)$.

If one of the edges in H is crossed, then at least three of the four vertices are involved in some crossing belong to N. There are exactly 10 pairs (y, x), where $y \in N$ and x is a crossing in D involving an edge incident with u. At least three of these pairs involve the same crossing; thus there are at most $(10 - 3)/2 + 1 = 4.5$ crossings in D, a contradiction. We may assume that none of the edges in H is crossed.

Thus, H is drawn in D with no crossings, and a is drawn in $\hat{D}$ in the face of $D[H]$ bounded by the 4-cycle $uwvx$ (see Figure 5(a)).

Let H' be the graph induced by $N \cup \{a\}$. The only crossings of H' in $\widehat{D}$ are those between the edges at a and the boundary of F_a in D (see Figure 5(b)). This restricted drawing of H' implies that the ends of a crossing pair of edges have exactly one element

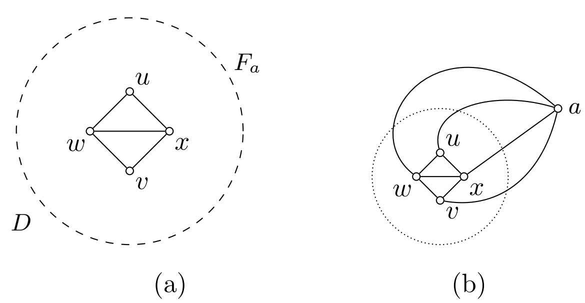

FIG. 5. *Subcase* 1.

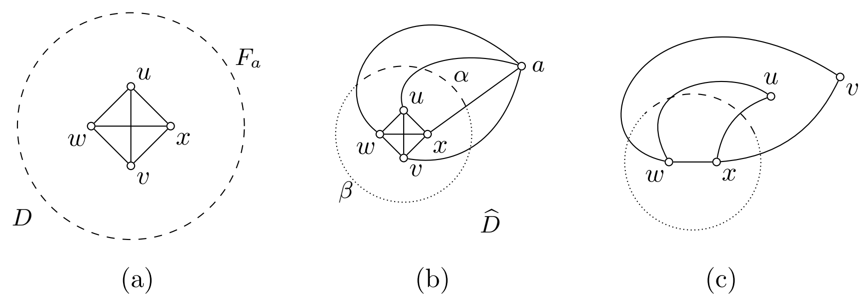

FIG. 6. *Subcase* 2.

in $\{u, v\}$ and exactly one element in $\{w, x\}$ and none of the two edges has both ends in N. However, this is not possible, as there are Four crossings involving edges incident to one of u or v, while there are six crossings involving edges incident to one of w or x.

Subcase 2. $D[H]$ is a K_4 with a crossing.

Suppose that uv and wx is the crossing pair in $D[H]$ and that c is the crossing between uv and wx. Following the same argument given in the previous case, it is easy to see that for every $y \in N$, there are exactly two crossings distinct from c involving edges at y; $cr(D) = 5$; if H' is the graph is induced by $N \cup \{a\}$, then its drawing $D[H']$ is isomorphic to the drawing of the cone of a crossing K_4, where the apex is drawn in the face bounded by the 4-cycle of the K_4 and the edges incident to the apex connect directly to the boundary of the 4-cycle (Figures 6(a) and (b)); and the only crossings of $\widehat{D}$ in H', distinct from c, are those between the edges at a and the boundary of F_a in D.

The restrictions on H' show that the ends of a crossing pair of edges distinct from uv and wx have exactly one element in $\{u, v\}$ and exactly one element in $\{w, x\}$ and none of the two edges has both ends in N.

The boundary walk of F_a contains a cycle C that in $\widehat{D}$ separates a from N. There are two internally disjoint subarcs α and β of $D[C]$ connecting the crossings between $D[C]$ and each of aw and ax. We label α and β so that α includes the crossing between $D[C]$ and au, and β includes the crossing between $D[C]$ and av. The restrictions imposed by the crossings in H' imply that all the neighbors of u not in N are contained in α, and likewise the neighbors of v not in N are contained in β.

We obtain a drawing of G with four crossings as follows. Redraw u in the place of a; join a to each of w and x using the corresponding edges from a to each of w and x. Draw the edges from u to its neighbors not in N without creating new crossings. Now redraw v near u in the face bounded by β and the two segments of the new uw, ux edges (Figure 6(c)). Connect v to each of w and x by following arcs near the new uw, ux edges. Connect v to the rest of its neighbors without creating new crossings. Since $cr(G) \geq 5$ and this drawing has four crossings, this is a contradiction.

In any case we obtained a contradiction. Thus $cr(CG) \geq k + 5$ when $k \geq 5$. ☐

5. Asymptotics for simple graphs. Finally, we try to understand the behavior of $f_s(k)$ when k is large. The important part is the increase of the crossing number after adding the apex; thus we define

$$\phi_s(k) = f_s(k) - k.$$

We have proved that $\phi(k) = f(k) - k \geq (\frac{1}{2}k)^{1/2}$. The term $k^{1/2}$ is asymptotically tight in the case when we allow multiple edges. However, it is unclear how large $\phi_s(k)$ is. This question is treated next.

THEOREM 8. $\phi_s(k) = O(k^{3/4})$.

Proof. Let us consider a positive integer k, and let n be the smallest integer such that $cr(K_n) \geq k$. Then $G = K_n$ has a crossing number at least k, and its cone is K_{n+1}.

To find an upper bound for $cr(K_{n+1})$ in terms of $cr(K_n)$, start with a drawing of K_n with $cr(K_n)$ crossings. Then clone a vertex; that is, place a new vertex very close to an original vertex and draw the new edges along the original edges. Each edge incident to the new vertex crosses $O(n^2)$ edges; thus the resulting drawing has $cr(K_n) + O(n^3)$ crossings. Therefore

$$\phi_s(k) \leq cr(K_{n+1}) - cr(K_n) \leq O(n^3).$$

It is known [16] that

$$\frac{3}{10}\binom{n}{4} \leq cr(K_n) \leq \frac{3}{8}\binom{n}{4}.$$

(The constant 3/10 in the lower bound obtained in [13] has been recently improved to 0.32025; see [16] for more information. The 3/8 factor follows from the existence of drawings of K_n having the same number of crossings as in the formula given in the Harary–Hill conjecture.) Then $\phi_s(k) = O(n^3) = O(k^{3/4})$. ∎

The *Harary–Hill conjecture* [14] states that

$$cr(K_n) = \begin{cases} \frac{1}{64}n(n-2)^2(n-4), & n \text{ is even,} \\ \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ is odd.} \end{cases}$$

PROPOSITION 9. *If the Harary–Hill conjecture holds, then*

$$\phi_s(k) \leq \sqrt{2}\, k^{3/4}(1 + o(1)).$$

Proof. As in the proof of Theorem 8, but with a slight twist for added precision, we take n such that $cr(K_{n-1}) < k \leq cr(K_n)$. We also take n_1 such that for $k_1 = k - cr(K_{n-1})$ we have $cr(K_{n_1-1}) < k_1 \leq cr(K_{n_1})$. Let $G = K_{n-1} \cup K_{n_1}$. Then $cr(G) = cr(K_{n-1}) + cr(K_{n_1}) \geq k$ and $cr(CG) = cr(K_n) + cr(K_{n_1+1})$. Therefore,

$$\begin{aligned} \phi_s(k) &\leq cr(K_n) + cr(K_{n_1+1}) - cr(K_{n-1}) - cr(K_{n_1}) \\ &\leq cr(K_n) - cr(K_{n-1}) + cr(K_{n_1+1}) - cr(K_{n_1-1}). \end{aligned}$$

By inserting the values for the crossing number from the Harary–Hill conjecture, we obtain (the calculation given is for odd n and odd n_1; it is similar when n or n_1 is even) the following:

$$cr(K_n) - cr(K_{n-1}) = \tfrac{1}{64}((n-1)^2(n-3)^2 - (n-1)(n-3)^2(n-5)) = \tfrac{1}{16}n^3(1 + o(1))$$

and

$$\begin{aligned} cr(K_{n_1+1}) - cr(K_{n_1-1}) &= \tfrac{1}{64}((n_1+1)(n_1-1)^2(n_1-3) \\ &\quad -(n_1-1)(n_1-3)^2(n_1-5)) \\ &= \tfrac{1}{8}n_1^3(1 + o(1)). \end{aligned}$$

Noticing that $k = \frac{1}{64}n^4(1 + o(1))$ and $k_1 = \frac{1}{64}n_1^4(1 + o(1)) = O(n^3)$ because $k_1 \leq cr(K_n) - cr(K_{n-1})$, we conclude that $n_1^3 = O(n^{9/4}) = o(k^{3/4})$ and henceforth

$$\phi_s(k) \leq \tfrac{1}{16}n^3(1 + o(1)) + \tfrac{1}{8}n_1^3(1 + o(1)) = \sqrt{2}\, k^{3/4}(1 + o(1)).$$ ∎

The above proof works even under a weaker hypothesis that $cr(K_n) = \alpha n^4 + \beta n^3(1 + o(1))$, where α and β are constants. This would imply that $\phi_s(k) = O(k^{3/4})$. Our conjecture is that Proposition 9 gives the precise asymptotics.

CONJECTURE 10. *$\phi_s(k) = \sqrt{2}\, k^{3/4}(1 + o(1))$.*

A reviewer of a preliminary version of this paper noted that this asymptotic is matched when the graph we are considering is dense.

Remark 11. Let G be a graph with n vertices, m edges, $cr(G) = k$ and such that $m \geq 4n$. If $m = \Omega(n^2)$, then $cr(CG) \geq k + \Omega(k^{3/4})$.

Proof. First we show that $cr(CG) \geq k + 4k/n$. Suppose not. Consider an optimal drawing of CG, and let D be its restriction to G. For $v \in V(G)$, let s_v be the number of crossings in D involving edges incident to v. For every $v \in V(G)$, s_v is less than $4k/n$, as otherwise, when we remove v and the edges that connect the apex of CG to the nonneighbors of v, we obtain a drawing of G with less than k crossings.

Since each crossing contributes to the value s_v of four vertices and D has at least k crossings, $4k \leq \sum_{v \in V(G)} s_v < n(4k/n)$, a contradiction. Thus $cr(CG) \geq k + 4k/n$.

Now, we use the crossing lemma [1] to find that $k = \Omega(m^3/n^2) = \Omega(n^4)$ and then obtain that $cr(CG) \geq k + 4k/n = k + \Omega(k^{3/4})$. ☐

Summary. To put the results of this paper into context, let us review the motivation behind this paper and suggest some directions for future work. The starting point of this paper was an attempt to understand Albertson's conjecture. The results of the paper (and their proofs) show that the crossing number behavior when adding an apex vertex is closely related to 1-page drawings, but the exact relationship is quite subtle.

If we restrict our problem to simple graphs, the family of complete graphs is (asymptotically) our best-known example for the minimal increase of the crossing number when the apex is added. This lead us to formulate Conjecture 10. Although very dense graphs, such as complete graphs, have fewer vertices than sparser graphs with the same crossing number and thus need fewer connections to be made from the apex to their vertices, their near-optimal drawings are far from 1-page drawings, and therefore more crossings are needed. A full understanding of this antinomy would shed new light on the Harary–Hill conjecture.

Finally, it is worth pointing out that neither an exact nor an approximation algorithm is known for computing the crossing number of graphs of bounded tree-width (Biedl et al. [7] recently found an approximation algorithm for graphs of bounded path-width). Adding an apex to a graph increases the tree-width of the graph by 1; thus understanding the crossing number of the cone is an important special case that would need to be understood before devising an algorithm for general graphs of bounded tree-width.

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